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THE GALERKIN METHOD AS APPLIED TO PROBLEMS IN VISCOELASTICITY

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Abstract—This paper establishes the convergence of the continuous-time Galerkin technique as applied to quasi-static, linear viscoelasticity.

INTRODUCTION

In recent years the finite element method has been successfully applied to boundary-value problems within the quasi-static, linear theory of viscoelasticity[1–5]. The finite element method, as applied in these circumstances, is a special case of what is now referred to as the continuous-time Galerkin technique. In this paper we establish the convergence of this technique. Our results are sufficiently general to include thermoviscoelastic bodies† under the influence of a prescribed time-dependent temperature field.

1. NOTATION

Throughout this paper B designates a (compact) properly regular[6, 7] region of three-dimensional Euclidean space, while $\mathscr{S}$ and $\mathscr{S}_*$ are complementary closed regular[7] subsurfaces of the boundary ∂B of B:

$$\partial B = \mathscr{S} \cup \mathscr{S}_*, \qquad \mathring{\mathscr{S}} \cap \mathring{\mathscr{S}}_* = \phi. \tag{1.1}$$

Let υ denote the inner product space (translation space) associated with Euclidean space; $\mathbf{u} \cdot \mathbf{v}$ is the inner product of $\mathbf{u}, \mathbf{v} \in \upsilon$. We use the term tensor as a synonym for "linear transformation from υ into υ." A tensor $\mathbf{A}$ is *symmetric* if $\mathbf{A} = \mathbf{A}^T$, *skew* if $\mathbf{A} = -\mathbf{A}^T$; here $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$. For convenience, we write

Sym = the space of symmetric tensors.

The inner product of two tensors $\mathbf{A}$ and $\mathbf{B}$ is defined by

$$\mathbf{A} \cdot \mathbf{B} = tr(\mathbf{A}\mathbf{B}^T), \tag{1.2}$$

where tr denotes the trace.

Given a vector field $\mathbf{u}$ on B, we write $\nabla\mathbf{u}$ for its generalized gradient [8] and

$$\hat{\nabla}\mathbf{u} = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \tag{1.3}$$

† E.g. thermorheologically simple viscoelastic bodies.

for its generalized symmetric gradient. Further,

$$W(B) = W_2^1(B) \tag{1.4}$$

is the Sobolev space consisting of all vector fields $\mathbf{u}$ on B such that both $\mathbf{u}$ and $\nabla\mathbf{u}$ belong to $L_2(B)$; the norm of $\mathbf{u} \in W(B)$ is, of course, defined by

$$\|\mathbf{u}\|^2_{W(B)} = \|\mathbf{u}\|^2_{L_2(B)} + \|\nabla\mathbf{u}\|^2_{L_2(B)}, \tag{1.5}$$

where

$$\|\mathbf{u}\|^2_{L_2(B)} = \int_B \mathbf{u} \cdot \mathbf{u}, \qquad \|\nabla\mathbf{u}\|^2_{L_2(B)} = \int_B \nabla\mathbf{u} \cdot \nabla\mathbf{u} \tag{1.6}$$

are the $L_2(B)$ norms of $\mathbf{u}$ and $\nabla\mathbf{u}$.

Given two tensor fields $\mathbf{A}$, $\mathbf{B} \in L_2(B)$ we write

$$\langle \mathbf{A}, \mathbf{B} \rangle = \int_B \mathbf{A} \cdot \mathbf{B}, \tag{1.7}$$

so that

$$\langle \mathbf{A}, \mathbf{A} \rangle = \|\mathbf{A}\|^2_{L_2(B)}. \tag{1.8}$$

A vector field $\mathbf{r}$ on B is a **rigid displacement field** if it admits the representation

$$\mathbf{r}(\mathbf{x}) = \mathbf{a} + \mathbf{W}[\mathbf{x} - \mathbf{x}_0], \tag{1.9}$$

where $\mathbf{a}$ is a vector, $\mathbf{W}$ a skew tensor, and $\mathbf{x}_0$ a point. As is well known, $\mathbf{r} \in W(B)$ is rigid if and only if†

$$\hat{\nabla}\mathbf{r} = \mathbf{0}. \tag{1.10}$$

Remark.‡ *If $\mathscr{S} \neq \phi$, and if $\mathbf{r}$ is a rigid displacement field that vanishes on $\mathscr{S}$, then $\mathbf{r} = \mathbf{0}$.*

We will frequently deal with functions $\Psi(\mathbf{x}, t)$ of position $\mathbf{x} \in B$ and time $t \in T$, where T is an interval of the reals, R. For such a function we write Ψ_t for the *field* on B defined by

$$\Psi_t(\mathbf{x}) = \Psi(\mathbf{x}, t). \tag{1.11}$$

Finally, a **rigid motion** is a vector field $\mathbf{r}$ on $B \times T$ with $\mathbf{r}_t$ a rigid displacement for each $t \in T$.

2. THE BOUNDARY-VALUE PROBLEM—WEAK SOLUTIONS

The fundamental system of field equations for a linear viscoelastic solid consists of the *equation of equilibrium*

$$\operatorname{div} \mathbf{S} + \mathbf{b} = \mathbf{0} \tag{2.1}$$

and the *constitutive relation*§

$$\mathbf{S}(\mathbf{x}, t) = \mathring{\mathbf{S}}(\mathbf{x}, t) + \mathbf{C}(\mathbf{x})\hat{\nabla}\mathbf{u}(\mathbf{x}, t) + \int_0^t \mathbf{K}(\mathbf{x}, t, \tau)\hat{\nabla}\mathbf{u}(\mathbf{x}, \tau)\, d\tau. \tag{2.2}$$

† For smooth fields this result is well known; for functions in $W(B)$ see, e.g. Fichera[6], p. 384 and Hlaváček and Nečas[9], Lemma II.1.

‡ See, e.g. Hlaváček and Nečas[9], Lemma II.3. (Note that since $\mathscr{S}$ is a regular subsurface of ∂B, if $\mathscr{S} \neq \phi$, then $\mathfrak{Z} \neq \phi$.)

§ In the usual quasi-static theory $\mathring{\mathbf{S}} = \mathbf{0}$ and $\mathbf{K}(\mathbf{x}, t, \tau) = \mathbf{K}(\mathbf{x}, t - \tau)$. The field $\mathring{\mathbf{S}}$ is the stress that would be present in the material if the strain $\hat{\nabla}\mathbf{u}$ were zero for $t \geq 0$. Its presence allows for the possibility of a prescribed non-zero strain history up to time $t = 0$. Also, the general theory presented here includes, as a special case, thermoviscoelastic materials in situations for which the temperature field is known *a priori*.

Here $\mathbf{u}$ is the displacement field, $\mathbf{S}$ is the stress field, $\mathring{\mathbf{S}}$ is the residual stress field, and $\mathbf{b}$ is the body force field. The fields $\mathbf{C}$ and $\mathbf{K}$ are material response functions; they describe, respectively, the instantaneous and the delayed response of the material. The field equations (2.1) and (2.2) must be satisfied at every $\mathbf{x} \in B$ and for all $t \in T$, where B is the region of space occupied by the body, and T is a (possibly infinite) time-interval of the form $[0, a)$.

To these field equations we adjoin the *boundary conditions*

$$\mathbf{u} = \mathbf{h} \quad \text{on} \quad \mathcal{S} \times T, \qquad \mathbf{Sn} = \mathbf{s} \quad \text{on} \quad \mathcal{S}_* \times T, \tag{2.3}$$

where $\mathbf{h}$ and $\mathbf{s}$ are, respectively, the prescribed surface displacement and surface traction, and $\mathbf{n}$ is the outward unit normal to ∂B.

The boundary-value problem under consideration consists in the following: *given*: $\mathbf{C}$, $\mathbf{K}$, $\mathbf{b}$, $\mathring{\mathbf{S}}$, $\mathbf{h}$, $\mathbf{s}$; *find*: fields $\mathbf{u}$ and $\mathbf{S}$ that satisfy (2.1–2.3). For convenience, we assume, once and for all, that:

(A_1) $\mathbf{C}(\mathbf{x})$ (for $\mathbf{x} \in B$) and $\mathbf{K}(\mathbf{x}, t, \tau)$ (for $\mathbf{x} \in B$ and $0 \le \tau \le t < \infty$) are linear transformations from Sym into Sym; $\mathbf{C}(\mathbf{x})$ is *symmetric* and *positive definite*,† that is,

$$\mathbf{A} \cdot \mathbf{C}(\mathbf{x})\mathbf{B} = \mathbf{B} \cdot \mathbf{C}(\mathbf{x})\mathbf{A} \tag{2.4}$$

for all $\mathbf{A}$, $\mathbf{B} \in$ Sym and there exists a constant $\kappa > 0$ such that

$$\mathbf{A} \cdot \mathbf{C}(\mathbf{x})\mathbf{A} \ge \kappa \mathbf{A} \cdot \mathbf{A} \tag{2.5}$$

for all $\mathbf{A} \in$ Sym; the mappings $\mathbf{x} \mapsto \mathbf{C}(\mathbf{x})$ and $(\mathbf{x}, t, \tau) \mapsto \mathbf{K}(\mathbf{x}, t, \tau)$ are continuous.

(A_2) $\mathring{\mathbf{S}} = \mathring{\mathbf{S}}^T$, and the mappings $t \mapsto \mathring{\mathbf{S}}_t$, $t \mapsto \mathbf{b}_t$, $t \mapsto \mathbf{h}_t$, and $t \mapsto \mathbf{s}_t$ on T have values in $L_2(B)$, $L_2(B)$, $L_2(\mathcal{S})$, and $L_2(\mathcal{S}_*)$, respectively, and (as L_2-valued mappings) are piecewise continuous.

(A_3) When $\mathcal{S} = \phi$ (so that $\mathcal{S}_* = \partial B$) the prescribed loads are in equilibrium; that is,

$$\int_{\partial B} \mathbf{s} \cdot \mathbf{r} + \int_B \mathbf{b} \cdot \mathbf{r} = 0 \tag{2.6}$$

for every rigid displacement $\mathbf{r}$.‡

Note that, by (2.5) and the continuity of $\mathbf{C}$ on B, there exists a constant $\kappa_1 \ge \kappa$ such that for every function $\mathbf{A}\colon B \to$ Sym belonging to $L_2(B)$

$$\kappa \|\mathbf{A}\|^2_{L_2(B)} \le \langle \mathbf{A}, \mathbf{CA} \rangle \le \kappa_1 \|\mathbf{A}\|^2_{L_2(B)}. \tag{2.7}$$

We call

$$\Phi = \{\boldsymbol{\varphi} \in W(B) \colon \boldsymbol{\varphi} = \mathbf{0} \text{ on } \mathcal{S}\} \tag{2.8}$$

the **variation space**; fields $\boldsymbol{\varphi} \in \Phi$ will be referred to as **variations**. It is important to note that each variation is a function of $\mathbf{x}$ alone.

Now let $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{S}(\mathbf{x}, t)$ constitute a sufficiently smooth solution to the boundary-value problem (2.1–2.3), and let $\boldsymbol{\varphi}(\mathbf{x})$ be a variation. By (A_1), (A_2) and (2.2), $\mathbf{S}$ is symmetric; therefore

$$(\operatorname{div} \mathbf{S}_t) \cdot \boldsymbol{\varphi} = \operatorname{div}(\mathbf{S}_t \boldsymbol{\varphi}) - \mathbf{S}_t \cdot \hat{\nabla} \boldsymbol{\varphi}, \tag{2.9}$$

† Coleman[10] has shown that $\mathbf{C}(\mathbf{x})$ positive semi-definite and symmetric is a consequence of the second law of thermodynamics. Gurtin and Herrera[11] have shown that $\mathbf{C}(\mathbf{x})$ positive definite and symmetric follows from the requirement that work be done to deform the body from an equilibrium state.

‡ This is equivalent to the usual force and moment balance equations for B (c.f., e.g. Gurtin[7], Theorem 18.3).

where we have used the notation (1.11). If we take the inner product of (2.1) with $\boldsymbol{\varphi}$, integrate over B, and use the divergence theorem in conjunction with $(2.3)_2$, (2.8), (2.9) and (1.7), we arrive at

$$\langle \mathbf{S}_t, \hat{\nabla}\boldsymbol{\varphi} \rangle = \int_{\mathscr{S}_*} \mathbf{s}_t \cdot \boldsymbol{\varphi} + \int_B \mathbf{b}_t \cdot \boldsymbol{\varphi}. \tag{2.10}$$

Thus, if we define

$$\mathscr{F}_t(\boldsymbol{\varphi}) = \int_{\mathscr{S}_*} \mathbf{s}_t \cdot \boldsymbol{\varphi} + \int_B \mathbf{b}_t \cdot \boldsymbol{\varphi} - \int_B \mathring{\mathbf{S}}_t \cdot \hat{\nabla}\boldsymbol{\varphi}, \tag{2.11}$$

then (2.2) and (2.10) imply that

$$\langle \mathbf{C}\hat{\nabla}\mathbf{u}_t, \hat{\nabla}\boldsymbol{\varphi} \rangle + \int_0^t \langle \mathbf{K}(t, \tau)\hat{\nabla}\mathbf{u}_\tau, \hat{\nabla}\boldsymbol{\varphi} \rangle \, d\tau = \mathscr{F}_t(\boldsymbol{\varphi}), \tag{2.12}$$

where we have written $\mathbf{K}(t, \tau)$ for the field on B with values $\mathbf{K}(t, \tau, \mathbf{x})$, so that

$$\langle \mathbf{K}(t, \tau)\hat{\nabla}\mathbf{u}_\tau, \hat{\nabla}\boldsymbol{\varphi} \rangle = \int_B [\mathbf{K}(t, \tau, \mathbf{x})\hat{\nabla}\mathbf{u}(\mathbf{x}, \tau)] \cdot \hat{\nabla}\boldsymbol{\varphi}(\mathbf{x}) \, d\mathbf{x}. \tag{2.13}$$

We have shown that every sufficiently smooth solution of (2.1–2.3) satisfies (2.12) for every t and every variation $\boldsymbol{\varphi}$. Conversely, (for sufficiently smooth data) it is not difficult to verify that every sufficiently smooth field $\mathbf{u}$ that satisfies $(2.3)_1$ and (2.12) for every t and every variation $\boldsymbol{\varphi}$ is a solution to the original problem (2.1–2.3). This should serve to motivate the following definitions.

By the **solution space** we mean the space $\mathscr{S}$ of all vector fields $\mathbf{u}$ on $B \times T$ such that $\mathbf{u}_t \in W(B)$ at each $t \in T$ and $t \to \hat{\nabla}\mathbf{u}_t$, as a mapping from T into $L_2(B)$, is piecewise continuous. A field $\mathbf{u} \in \mathscr{S}$ that satisfies

$$\mathbf{u} = \mathbf{h} \quad \text{on} \quad \mathscr{S} \times T \tag{2.14}$$

is **kinematically admissible**. A **weak solution** is a kinematically admissible field $\mathbf{u}$ that satisfies (2.12) for every $t \in T$ and every variation $\boldsymbol{\varphi}$.

3. THE CONTINUOUS TIME GALERKIN APPROXIMATION

Let Φ_N be an N-dimensional subspace of Φ, let $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_N$ be a basis for Φ_N, and let $\mathbf{h}^N \in \mathscr{S}$. We consider approximations of the form

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{h}^N(\mathbf{x}, t) + \sum_{n=1}^{N} \alpha_n(t)\boldsymbol{\varphi}_n(\mathbf{x}). \tag{3.1}$$

In applications $\{\boldsymbol{\varphi}_n\}$ and $\mathbf{h}^N$ are prescribed fields; the $\boldsymbol{\varphi}_n$ are basis functions, e.g. for the finite element method, while $\mathbf{h}^N$ is chosen to approximate $\mathbf{h}$ on $\mathscr{S} \times T$. Indeed, since $\Phi^N \subset \Phi$, each $\boldsymbol{\varphi}_n = \mathbf{0}$ on $\mathscr{S}$, and (3.1) implies that

$$\mathbf{v} = \mathbf{h}^N \quad \text{on} \quad \mathscr{S} \times T. \tag{3.2}$$

When $\mathscr{S}$ is empty we omit the function $\mathbf{h}^N$ and consider approximate solutions of the form

$$\mathbf{v}(\mathbf{x}, t) = \sum_{n=1}^{N} \alpha_n(t)\boldsymbol{\varphi}_n(x). \tag{3.3}$$

Thus let

$$\mathscr{G}_N = \mathscr{G}(\mathbf{h}^N, \Phi_N) = \{\mathbf{v} \in \mathscr{S}: \mathbf{v} \text{ has the form (3.1) (or (3.3) if } \mathscr{s} = \phi)\}. \tag{3.4}$$

We call $\mathscr{G}_N$ an **approximation space** of dimension N. By a **continuous-time Galerkin solution** for $\mathscr{G}_N$ we mean a function $\mathbf{v} \in \mathscr{G}_N$ that satisfies†

$$\langle \mathbf{C}\hat{\nabla}\mathbf{v}_t, \hat{\nabla}\boldsymbol{\varphi}\rangle + \int_0^t \langle \mathbf{K}(t, \tau)\hat{\nabla}\mathbf{v}_\tau, \hat{\nabla}\boldsymbol{\varphi}\rangle \, \mathrm{d}\tau = \mathscr{F}_t(\boldsymbol{\varphi}) \tag{3.5}$$

for every $t \in T$ and every $\boldsymbol{\varphi} \in \Phi_N$.

Let $\mathbf{C}$ and $\mathbf{K}(t, \tau)$ be the $N \times N$ matrices with entries

$$\begin{aligned} C_{mn} &= \langle \mathbf{C}\hat{\nabla}\boldsymbol{\varphi}_n, \hat{\nabla}\boldsymbol{\varphi}_m\rangle, \\ K_{mn}(t, \tau) &= \langle \mathbf{K}(t, \tau)\hat{\nabla}\boldsymbol{\varphi}_n, \hat{\nabla}\boldsymbol{\varphi}_m\rangle, \end{aligned} \tag{3.6}$$

and let $\boldsymbol{\alpha}(t)$ and $\mathbf{f}(t)$ be the $N \times 1$ column vectors with entries $\alpha_n(t)$ and

$$f_n(t) = \mathscr{F}_t(\boldsymbol{\varphi}_n) - \langle \mathbf{C}\hat{\nabla}\mathbf{h}_t^N, \hat{\nabla}\boldsymbol{\varphi}_n\rangle - \int_0^t \langle \mathbf{K}(t, \tau)\hat{\nabla}\mathbf{h}_\tau^N, \hat{\nabla}\boldsymbol{\varphi}_n\rangle \, \mathrm{d}\tau. \tag{3.7}$$

It then follows that (3.5) is equivalent to the integral equation

$$\mathbf{C}\boldsymbol{\alpha}(t) + \int_0^t \mathbf{K}(t, \tau)\boldsymbol{\alpha}(\tau) \, \mathrm{d}\tau = \mathbf{f}(t). \tag{3.8}$$

Existence theorem. *If $\mathscr{s} \neq \phi$, a unique continuous-time Galerkin solution for $\mathscr{G}_N$ exists. If $\mathscr{s} = \phi$, a solution always exists, but need not be unique; however, any two solutions differ at most by a rigid motion.*

We postpone, until Section 4, the proofs of both this and the next theorem.

We now consider a sequence $\{\mathscr{G}_N\}$, where each $\mathscr{G}_N = \mathscr{G}(\mathbf{h}^N, \Phi_N)$ is an approximation space of dimension N. We say that $\{\mathscr{G}_N\}$ is **complete** if given any kinetically admissible field $\mathbf{k}$ there exists a sequence $\{\mathbf{g}^N\}$ with $\mathbf{g}^N \in \mathscr{G}_N$ such that

$$\|\mathbf{k}_t - \mathbf{g}_t^N\|_{W(B)} \to 0 \quad \text{as} \quad N \to \infty \tag{3.9}$$

uniformly for t in any bounded subinterval of T. When this is the case $\{\mathbf{g}^N\}$ is an **approximating sequence** for $\mathbf{k}$.

Convergence theorem. *Let $\mathbf{u}$ be a weak solution to the boundary-value problem, and, for each N, let $\mathbf{u}^N$ be a continuous-time Galerkin solution for $\mathscr{G}_N$. Assume that the sequence $\{\mathscr{G}_N\}$ is complete. Then there exists a sequence $\{\mathbf{r}^N\}$ of rigid motions such that*

$$\|\mathbf{u}_t - \mathbf{u}_t^N - \mathbf{r}_t^N\|_{W(B)} \to 0 \quad \text{as} \quad N \to \infty \tag{3.10}$$

for all $t \in T$. Moreover, $\mathbf{r}^N$ may be set equal to zero when $\mathscr{s} \neq \phi$.‡

† The variational principle established by Gurtin ([12], eq. (3.12)) for the classical quasi-static problem, when applied in the usual manner to an approximate solution of the form (3.1), leads to the continuous-time Galerkin approximation (3.5).

‡ We could set $\mathbf{r}^N = 0$ when $\mathscr{s} = \phi$ if both $\mathbf{u}$ and the basis functions $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_N$ for each $\mathscr{G}_N$ were normalized (cf. (4.31)). A normalization of this type is utilized by Chou[13] to establish a convergence theorem appropriate to elastostatics.

4. ANALYSIS

Let

$$\mathcal{N} = \text{null space of } \mathbf{C},$$

$$\mathcal{R} = \text{range of } \mathbf{C}.$$

Lemma 1. (i) $\mathbf{C}$ *is symmetric.*
(ii) $\mathcal{R} = \mathcal{N}^{\perp}$.
(iii) $\mathcal{N} = \{\boldsymbol{\lambda} \in R^N : \sum_n \lambda_n \boldsymbol{\varphi}_n$ *is rigid*$\}$.
(iv) *The range of* $\mathbf{K}(t, \tau)$ *lies in* $\mathcal{N}^{\perp}$.
(v) $\mathcal{N}$ *lies in the null space of* $\mathbf{K}(t, \tau)$.
(vi) $\mathbf{f}(t) \in \mathcal{N}^{\perp}$ *whenever* $\mathscr{s} = \phi$.

Proof. Assertion (i) follows from (2.4) and $(3.6)_1$; and (i), in turn, implies (ii). To establish (iii) assume that

$$\sum_n C_{mn} \lambda_n = 0. \tag{4.1}$$

Then, letting

$$\mathbf{w} = \sum_n \lambda_n \boldsymbol{\varphi}_n, \tag{4.2}$$

we conclude, using $(3.6)_1$, that

$$0 = \sum_{m,n} C_{mn} \lambda_m \lambda_n = \langle \mathbf{C}\hat{\nabla}\mathbf{w}, \hat{\nabla}\mathbf{w} \rangle. \tag{4.3}$$

Thus by (2.5),

$$\hat{\nabla}\mathbf{w} = \mathbf{0} \tag{4.4}$$

and $\mathbf{w}$ is rigid. Conversely, assume that $\mathbf{w}$, given by (4.2), is rigid. Then, by $(3.6)_1$ and (4.4),

$$\sum_n C_{mn} \lambda_n = \langle \mathbf{C}\hat{\nabla}\mathbf{w}, \hat{\nabla}\boldsymbol{\varphi}_m \rangle = 0, \tag{4.5}$$

so that $\boldsymbol{\lambda} \in \mathcal{N}$. Thus (iii) holds.

Similarly, if $\boldsymbol{\lambda} \in \mathcal{N}$, and if $\mathbf{w}$ is the rigid displacement (4.2), then $(3.6)_2$ and (4.4) imply that

$$\sum_m \lambda_m K_{mn}(t, \tau) = 0, \qquad \sum_n K_{mn}(t, \tau)\lambda_n = 0, \tag{4.6}$$

and these results yield (iv) and (v).

Again, choose $\boldsymbol{\lambda} \in \mathcal{N}$ and let $\mathbf{w}$ be given by (4.2). Then, if $\mathscr{s} = \phi$, (2.11), (3.7), (4.4) and (2.6) imply that

$$\sum_n \lambda_n f_n(t) = \int_{\partial B} \mathbf{s}_t \cdot \mathbf{w} + \int_B \mathbf{b}_t \cdot \mathbf{w} = \mathbf{0}, \tag{4.7}$$

so that $\mathbf{f}(t) \in \mathcal{N}^{\perp}$. This completes the proof of Lemma 1.

Proof of the existence theorem. By (2.11), (A_1), (A_2), the fact that $\mathbf{h}^N \in \mathscr{S}$, and the definition of the solution space $\mathscr{S}$, the function $\mathbf{f}$ defined by (3.7) is piecewise continuous on T. Thus, if a solution $\boldsymbol{\alpha}$ of (3.8) exists, then $\boldsymbol{\alpha}$ must also be piecewise continuous on T; and, since $\mathbf{h}^N \in \mathscr{S}$ and each $\boldsymbol{\varphi}_n \in W(B)$, it is clear that the field $\mathbf{v}$, defined by (3.1) (or (3.3)), will be a member of the solution space $\mathcal{S}$ and hence of $\mathcal{G}_N$. Therefore, to establish the existence of a continuous-time Galerkin solution $\mathbf{v}$ for $\mathcal{G}_N$, it suffices to establish the existence of a solution $\boldsymbol{\alpha}$ of (3.8).

Assume first that $\mathscr{s} \neq \phi$. It suffices to show that $\mathbf{C}^{-1}$ exists, for then (3.8) reduces to an integral equation for $\boldsymbol{\alpha}$. By (iii), if $\mathbf{C}\boldsymbol{\lambda} = \mathbf{0}$, then $\mathbf{w}$, defined by (4.2), must be rigid. But $\mathbf{w}$ must vanish on $\mathscr{s}$, since each $\boldsymbol{\varphi}_n$ has this property; thus, in view of the remark made in Section 1, $\mathbf{w} = \mathbf{0}$, and, as $\{\boldsymbol{\varphi}_n\}$ is a basis for Φ_N, this implies that each λ_n must vanish. Thus in this instance $\mathcal{N} = \{\mathbf{0}\}$ and $\mathbf{C}$ is invertible.

Assume next that $\mathscr{s} = \phi$. It follows from (ii) that $\tilde{\mathbf{C}}: \mathscr{R} \to \mathscr{R}$, the restriction of $\mathbf{C}$ to $\mathscr{R}$, is invertible. Let $\tilde{\mathbf{K}}(t, \tau)$ denote the restriction of $\mathbf{K}(t, \tau)$ to $\mathscr{R}$. By (ii) and (iv), $\tilde{\mathbf{K}}(t, \tau): \mathscr{R} \to \mathscr{R}$. It therefore follows from (vi) that the integral equation

$$\tilde{\mathbf{C}}\boldsymbol{\alpha}(t) + \int_0^t \tilde{\mathbf{K}}(t, \tau)\boldsymbol{\alpha}(\tau)\,d\tau = \mathbf{f}(t)$$

has a unique solution $\boldsymbol{\alpha}: T \to \mathscr{R}$. Trivially, $\boldsymbol{\alpha}$ is also a solution of our original equation (3.8).

We have only to show that when $\mathscr{s} = \phi$ any two solutions of (3.8) differ by a rigid motion. Thus let $\boldsymbol{\alpha}$ denote the difference between two such solutions, so that

$$\mathbf{C}\boldsymbol{\alpha}(t) + \int_0^t \mathbf{K}(t, \tau)\boldsymbol{\alpha}(\tau)\,d\tau = \mathbf{0}. \tag{4.8}$$

Let

$$\boldsymbol{\alpha}(t) = \boldsymbol{\lambda}(t) + \boldsymbol{\beta}(t), \qquad \boldsymbol{\lambda}(t) \in \mathcal{N}, \qquad \boldsymbol{\beta}(t) \in \mathcal{N}^{\perp}. \tag{4.9}$$

Then, since $\mathcal{N}$ is the null space of $\mathbf{C}$, (ii), (v), (4.8), and (4.9) imply that $\boldsymbol{\beta}$ is the unique solution of

$$\tilde{\mathbf{C}}\boldsymbol{\beta}(t) + \int_0^t \tilde{\mathbf{K}}(t, \tau)\boldsymbol{\beta}(\tau)\,d\tau = \mathbf{0}. \tag{4.10}$$

Thus $\boldsymbol{\beta} = \mathbf{0}$, and we conclude from (4.9) and (iii) that

$$\mathbf{w}(\mathbf{x}, t) = \sum_n \alpha_n(t)\boldsymbol{\varphi}_n(\mathbf{x}) \tag{4.11}$$

is a rigid motion. This completes the proof.

Lemma 2. *Let α, β, γ: $T \to [0, \infty)$ with α and β piecewise continuous and γ monotone increasing. Assume that*

$$\alpha(t) \le \beta(t) + \gamma(t) \int_0^t \alpha(\sigma)\,d\sigma \tag{4.12}$$

for all $t \in T$. Then

$$\alpha(t) \le \beta(t) + \gamma(t) \int_0^t e^{(t-\sigma)\gamma(t)}\beta(\sigma)\,d\sigma \tag{4.13}$$

for all $t \in T$.

Proof. Choose $t \in T$ arbitrarily. Since γ is monotone increasing,

$$\alpha(\tau) \le \beta(\tau) + \gamma(t) \int_0^\tau \alpha(\sigma)\,d\sigma \tag{4.14}$$

for all $\tau \in [0, t]$, and we may conclude from Gronwall's inequality† that

$$\alpha(\tau) \leq \beta(\tau) + \gamma(t) \int_0^\tau e^{(\tau-\sigma)\gamma(t)} \beta(\sigma)\, d\sigma \tag{4.15}$$

for all $\tau \in [0, t]$. In particular, if we take $\tau = t$ we arrive at (4.13).

Let D: $W(B) \times W(B) \to R$ be defined by

$$D(\mathbf{v}, \mathbf{w}) = \langle \mathbf{C}\hat{\nabla}\mathbf{v}, \hat{\nabla}\mathbf{w} \rangle. \tag{4.16}$$

Then, as is clear from (A_1), D is symmetric, bilinear, and positive semi-definite. In fact, if we define

$$D(\mathbf{v}) = D(\mathbf{v}, \mathbf{v}), \tag{4.17}$$

then (2.7) implies that

$$\kappa \|\hat{\nabla}\mathbf{v}\|^2_{L_2(B)} \leq D(\mathbf{v}) \leq \kappa_1 \|\hat{\nabla}\mathbf{v}\|^2_{L_2(B)} \tag{4.18}$$

for every $\mathbf{v} \in W(B)$. Further, (A_1) and the definition of the solution space $\mathscr{S}$ yield the following implication:

$$\mathbf{w} \in \mathscr{S} \Rightarrow t \to D(\mathbf{w}_t) \textit{ is piecewise continuous.} \tag{4.19}$$

Lemma 3. *Let* $\mathbf{w} \in \mathscr{S}$, $\boldsymbol{\varphi} \in \Phi$. *Then given any* $\alpha > 0$

$$2\int_0^t \langle \mathbf{K}(t, \tau)\hat{\nabla}\mathbf{w}_\tau, \hat{\nabla}\boldsymbol{\varphi} \rangle\, d\tau \leq m(t)\left[\frac{t}{\alpha} D(\boldsymbol{\varphi}) + \alpha \int_0^t D(\mathbf{w}_\tau)\, d\tau\right], \tag{4.20}$$

where

$$m(t) = \frac{1}{\kappa} \sup_{(\mathbf{x}, \tau) \in B \times [0, t]} |\mathbf{K}(\mathbf{x}, t, \tau)|. \tag{4.21}$$

Proof. Let $\|\cdot\| = \|\cdot\|_{L_2(B)}$. Then

$$\begin{aligned} 2\int_0^t \langle \mathbf{K}(t, \tau)\hat{\nabla}\mathbf{w}_\tau, \hat{\nabla}\boldsymbol{\varphi} \rangle\, d\tau &\leq 2\kappa m(t) \int_0^t \|\hat{\nabla}\mathbf{w}_\tau\| \|\hat{\nabla}\boldsymbol{\varphi}\|\, d\tau, \\ &\leq \kappa m(t) \int_0^t \left[\alpha \|\hat{\nabla}\mathbf{w}_\tau\|^2 + \frac{1}{\alpha} \|\hat{\nabla}\boldsymbol{\varphi}\|^2\right] d\tau, \end{aligned} \tag{4.22}$$

where we have used the inequality

$$2ab \leq \alpha a^2 + \frac{1}{\alpha} b^2.$$

The desired result (4.20) follows from (4.18) and (4.22).

Lemma 4. *Let* $\mathbf{u}$ *be a weak solution, and let* $\mathbf{v}$ *be a continuous-time Galerkin solution for* $\mathscr{G}_N$. *Further, let* $\mathbf{g} \in \mathscr{G}_N$ *be arbitrary, and let*

$$\mathbf{p} = \mathbf{u} - \mathbf{g}, \qquad \mathbf{q} = \mathbf{v} - \mathbf{g}. \tag{4.23}$$

† See, e.g. Reid[14], p. 13.

Then given any $t_0 \in T$ there exists a number $c_0(t_0)$ such that

$$D(\mathbf{q}_t) \leq c_0(t_0) \sup_{\tau \in [0, t_0]} D(\mathbf{p}_\tau) \tag{4.24†}$$

for all $t \in [0, t_0]$.

Proof. Since $\Phi_N \subset \Phi$, it follows from (2.12), (3.5), (4.16) and (4.23) that

$$D(\mathbf{q}_t, \boldsymbol{\varphi}) + \int_0^t \langle \mathbf{K}(t, \tau)\hat{\nabla}\mathbf{q}_\tau, \hat{\nabla}\boldsymbol{\varphi}\rangle \, d\tau = D(\mathbf{p}_t, \boldsymbol{\varphi}) + \int_0^t \langle \mathbf{K}(t, \tau)\hat{\nabla}\mathbf{p}_\tau, \hat{\nabla}\boldsymbol{\varphi}\rangle \, d\tau \tag{4.25}$$

for all $\boldsymbol{\varphi} \in \Phi_N$. Since both $\mathbf{v}$ and $\mathbf{g}$ belong to $\mathscr{G}_N$, it is clear from (3.4) and $(4.23)_2$ that $\mathbf{q}_t \in \Phi_N$. If we take $\boldsymbol{\varphi}$ in (4.25) equal to $\mathbf{q}_t$, and use the inequality

$$2D(\mathbf{p}_t, \mathbf{q}_t) \leq D(\mathbf{p}_t) + D(\mathbf{q}_t) \tag{4.26}$$

and (4.20) (first with $\mathbf{w} = \mathbf{q}$, $\boldsymbol{\varphi} = \mathbf{q}_t$, then with $\mathbf{w} = \mathbf{p}$, $\boldsymbol{\varphi} = \mathbf{q}_t$), we arrive at the inequality

$$D(\mathbf{q}_t) \leq D(\mathbf{p}_t) + m(t)\left[\frac{2t}{\alpha} D(\mathbf{q}_t) + \alpha \int_0^t [D(\mathbf{q}_\tau) + D(\mathbf{p}_\tau)] \, d\tau\right], \tag{4.27}$$

where $\alpha > 0$ is arbitrary. Since we are at liberty to let α in (4.27) depend on t, we can take $\alpha = 4tm(t)$ for $tm(t) \neq 0$. Then, letting

$$\begin{aligned} \gamma(t) &= 8tm^2(t), \\ \beta(t) &= 2D(\mathbf{p}_t) + \gamma(t) \int_0^t D(\mathbf{p}_\tau) \, d\tau, \end{aligned} \tag{4.28}$$

we conclude that

$$D(\mathbf{q}_t) \leq \beta(t) + \gamma(t) \int_0^t D(\mathbf{q}_\tau) \, d\tau, \tag{4.29}$$

for $tm(t) \neq 0$. By (4.27), (4.29) also holds when $tm(t) = 0$; hence (4.29) holds for all $t \in T$. It is clear from (4.21) and $(4.28)_1$ that γ is non-negative and monotone increasing. Further, since D is positive semi-definite, $\alpha(t) = D(\mathbf{q}_t) \geq 0$ and $\beta(t) \geq 0$, while (4.19) implies that α and β are piecewise continuous on T. Thus we may use Lemma 2 to arrive at

$$D(\mathbf{q}_t) \leq \beta(t) + \gamma(t) \int_0^t e^{(t-\tau)\gamma(t)} \beta(\tau) \, d\tau. \tag{4.30}$$

Clearly, (4.30) and (4.28) imply (4.24), and the proof is complete.

A field $\boldsymbol{\varphi} \in W(B)$ is **normalized** if

$$\int_B \boldsymbol{\varphi} = \mathbf{0}, \qquad \int_B (\nabla\boldsymbol{\varphi} - \nabla\boldsymbol{\varphi}^T) = \mathbf{0}. \tag{4.31}$$

It is not difficult to show that if $\boldsymbol{\varphi}$ is rigid and normalized, then $\boldsymbol{\varphi} = \mathbf{0}$. Further, for any $\boldsymbol{\varphi} \in W(B)$ there exists a unique rigid displacement $\mathbf{r}$ such that $\boldsymbol{\varphi} + \mathbf{r}$ is normalized; in this case we say that $\mathbf{r}$ **normalizes** $\boldsymbol{\varphi}$.

† This inequality is the crucial step in the proof of convergence. In this regard see Aubin[15], p. 132, Douglas and Dupont[16], and Price and Varga[17], who utilize an analysis of this type for parabolic problems.

Lemma 5. (Korn's Inequality).† *There exists a positive constant* c *such that*

$$\|\boldsymbol{\varphi}\|^2_{W(B)} \leq cD(\boldsymbol{\varphi}) \tag{4.32}$$

for each $\boldsymbol{\varphi} \in \Phi$, *provided* $\boldsymbol{\varphi}$ *is normalized if* $\mathscr{S} = \phi$.

Proof of the convergence theorem. Let $\{\mathbf{g}^N\}$ be an approximating sequence for $\mathbf{u}$ (such a sequence exists since $\{\mathscr{G}_N\}$ is complete), and let

$$\mathbf{p}^N = \mathbf{u} - \mathbf{g}^N, \qquad \mathbf{q}^N = \mathbf{u}^N - \mathbf{g}^N. \tag{4.33}$$

Then (3.9) with $\mathbf{k} = \mathbf{u}$ asserts that

$$\|\mathbf{p}_t^N\|_{W(B)} \to 0 \quad \text{as} \quad N \to \infty \tag{4.34}$$

uniformly on any bounded interval of T. Thus, since

$$\|\hat{\nabla}\mathbf{p}_t^N\|_{L_2(B)} \leq \|\mathbf{p}_t^N\|_{W(B)}, \tag{4.35}$$

we conclude from (4.18) and Lemma 4 that

$$D(\mathbf{q}_t^N) \to 0 \quad \text{as} \quad N \to \infty. \tag{4.36}$$

Let $\mathbf{r}^N$ be the rigid motion with the following property: $\mathbf{r}^N = 0$ if $\mathscr{S} \neq \phi$; $\mathbf{r}_t^N$ normalizes $\mathbf{q}_t^N$ at each $t \in T$ if $\mathscr{S} = \phi$. By the remarks made in the paragraph containing (4.25), it is clear that $\mathbf{q}_t^N \in \Phi$. Thus we conclude from Korn's inequality (Lemma 5) and (4.36) that

$$\|\mathbf{q}_t^N + \mathbf{r}_t^N\|_{W(B)} \to 0 \quad \text{as} \quad N \to \infty. \tag{4.37}$$

Here we have used the fact that, because $\mathbf{r}^N$ is rigid, $D(\mathbf{q}_t^N + \mathbf{r}_t^N) = D(\mathbf{q}_t^N)$. Next, by (4.33),

$$\|\mathbf{u}_t - \mathbf{u}_t^N - \mathbf{r}_t^N\|_{W(B)} = \|\mathbf{p}_t^N - \mathbf{q}_t^N - \mathbf{r}_t^N\|_{W(B)} \leq \|\mathbf{p}_t^N\|_{W(B)} + \|\mathbf{q}_t^N + \mathbf{r}_t^N\|_{W(B)}, \tag{4.38}$$

and (4.34), (4.37) and (4.38) imply (3.10). This completes the proof of the convergence theorem.

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† See, e.g. Hlaváček and Nečas[9], Theorem II.11. These authors show that (4.32) holds provided the underlying space V of functions $\boldsymbol{\varphi}$ is contained in $W(B)$ and has the following property: $\boldsymbol{\varphi} \in V$ and $\boldsymbol{\varphi}$ rigid implies $\boldsymbol{\varphi} = \mathbf{0}$. The underlying space V in our statement of the theorem clearly has this property. Indeed, when $\mathscr{S} \neq \phi$ each $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ satisfies $\boldsymbol{\varphi} = \mathbf{0}$ on $\mathscr{S}$; by the remark made in Section 1, the only rigid $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ with this property is $\boldsymbol{\varphi} = \mathbf{0}$. On the other hand, when $\mathscr{S} = \phi$ we require that $\boldsymbol{\varphi}$ be normalized; again, the only normalized, rigid $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ is $\boldsymbol{\varphi} = \mathbf{0}$.

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Абстракт — Метод финитного элемента последних лет успешно применялся к проблемам граничного значения в пределах квазистатической теории вазкоупругости. Метод финитного элемента, применяемый при этих условиях, является специальным случаем, на который теперь ссылаются как метод непрерывного времени Галеркина. В настоящей работе устанавливают конвергенцию этого метода. Полученные результаты достаточно общие для включения термовязкоупругих твердых тел под влиянием предписанного, зависимого от времени, температурного поля.